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# Geometric phases, symmetries of dynamical invariants and exact solution of the Schrödinger equation

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## Abstract

We introduce the notion of the geometrically equivalent quantum systems (GEQs) as quantum systems that lead to the same geometric phases for a given complete set of initial state vectors. We give a characterization of the GEQs. These systems have a common dynamical invariant, and their Hamiltonians and evolution operators are related by symmetry transformations of the invariant. If the invariant is  $T$ -periodic, the corresponding class of GEQs includes a system with a  $T$ -periodic Hamiltonian. We apply our general results to study the classes of GEQs that include a system with a cranked Hamiltonian  $H(t) = e^{-iKt} H_0 e^{iKt}$ . We show that the cranking operator  $K$  also belongs to this class. Hence, in spite of the fact that it is time independent, it leads to nontrivial cyclic evolutions and geometric phases. Our analysis allows for an explicit construction of a complete set of nonstationary cyclic states of any time-independent simple harmonic oscillator. The period of these cyclic states is half the characteristic period of the oscillator.

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## 1. Introduction

The history of modern physics is full of happy surprises. Among the latest of these is the discovery of Berry's adiabatic geometric phase [1–5]. This discovery has led physicists to reconsider a number of fundamental as well as practical aspects of quantum mechanics. The importance of Berry's findings and their impact on various areas of physics have naturally resulted in interest in the generalizations of geometric phases. One of the most significant contributions in this direction is the nonadiabatic generalization of Berry's phase due to Aharonov and Anandan [6]. This generalization employs a geometric picture of quantum dynamics and shows that the nonadiabatic geometric phase can be defined for any closed curve in the space of (pure) quantum states. Moreover, the geometric phase associated with the

evolution of a pure state only depends on the path traced by the evolving state in the state space. In other words, different Hamiltonians leading to the same path define the same geometric phase<sup>1</sup>. The purpose of this article is to address the problem of the characterization of all the quantum systems (with a common Hilbert space) that lead to the same geometric phases for a complete set of initial state vectors. We will term such systems '*geometrically equivalent*', and determine the Hamiltonian  $\tilde{H}(t)$  for a general quantum system that is geometrically equivalent to a given system with Hamiltonian  $H(t)$ .

Our main results are:

- (1) This problem is directly linked with the symmetries of dynamical invariants associated with the Hamiltonian  $H(t)$ .
- (2) The evolution operator  $\tilde{U}(t)$  for  $\tilde{H}(t)$  can be obtained from the evolution operator  $U(t)$  for  $H(t)$ .
- (3) Each class of geometrically equivalent quantum systems (GEQSs) includes a system whose dynamical and geometric phases cancel each other, i.e. their total phase is unity<sup>2</sup>. In particular, if the corresponding dynamical invariant is  $T$ -periodic, the class includes a system with a  $T$ -periodic Hamiltonian  $H_*(t)$ . Moreover, the evolution operator  $U_*(t)$  corresponding to  $H_*(t)$  satisfies  $U_*(T) = 1$ , i.e. the system has an evolution loop [7] of period  $T$ .
- (4) For the cranked Hamiltonians,  $H(t) = e^{-iKt} H_0 e^{iKt}$ , the cranking operator  $K$  is geometrically equivalent to  $H(t)$ . Therefore, although it is time independent, it leads to nontrivial cyclic evolutions and geometric phases.
- (5) Any time-independent simple harmonic oscillator admits a periodic dynamical invariant whose period is half the characteristic period of the oscillator. We construct this invariant and the associated exact cyclic states, and show that they acquire nontrivial cyclic geometric phases.

The organization of the paper is as follows. In section 2, we present a brief review of dynamical invariants and their relationship with geometric phases. In section 3, we address the characterization of the GEQS and develop a new approach to identify a class of exactly solvable quantum systems. In section 4, we consider the quantum systems which are geometrically equivalent to a cranked Hamiltonian. In section 5, we apply our general results to study a complete set of nonstationary cyclic states of a time-independent simple harmonic oscillator. In section 6, we present our concluding remarks.

## 2. Dynamical invariants and geometric phases

Consider a time-dependent Hamiltonian  $H(t)$  with the following properties:

- (1)  $H(t)$  is obtained from a Hermitian parametric Hamiltonian  $H[R]$  as  $H(t) = H[R(t)]$ , where  $R$  stands for  $(R^1, R^2, \dots, R^d)$ ,  $R^i$  are real parameters denoting the coordinates of points of a parameter manifold  $M$  and  $R(t)$  describes a smooth curve lying in  $M$ .
- (2) The spectrum of  $H[R]$  is discrete for all  $R \in M$ .
- (3) In local patches of  $M$ , the eigenvalues  $E_n[R]$  of  $H[R]$  are smooth functions of  $R$ .
- (4) The curve  $R(t)$  defining  $H(t)$  is such that during the evolution of the system, i.e. for all  $t \in [0, \tau]$ ,  $E_m[R(t)] = E_n[R(t)]$  if and only if  $m = n$ . In particular, the degree of

<sup>1</sup> This is usually demonstrated by showing that a shift of the Hamiltonian  $H(t)$  by a multiple of the identity operator  $I$ , i.e.  $H(t) \rightarrow H'(t) = H(t) + f(t)I$ , leaves the geometric phases invariant [4]. This shift is related to a global phase transformation of the Hilbert space. It is in a sense a trivial kinematic symmetry transformation [5].

<sup>2</sup> Here we mean the phases associated with the eigenvectors of the common dynamical invariant.

degeneracy of the energy eigenvalues does not depend on time, and no level-crossings occur.

- (5) In local patches of  $M$ , there is a set of orthonormal basis vectors  $|n, a; R\rangle$  of  $H[R]$  that are smooth functions of  $R$ .

Here  $a$  is a degeneracy label taking its values in  $\{1, 2, \dots, \mathcal{N}\}$  and  $\mathcal{N}$  denotes the degree of degeneracy of  $E_n(t) := E_n[R(t)]$ .

By definition, a dynamical invariant  $I(t)$  of the Hamiltonian  $H(t)$  is a nontrivial solution of the Liouville–von-Neumann equation<sup>3</sup> [8, 9]

$$\frac{d}{dt} I(t) = i[I(t), H(t)]. \tag{1}$$

Consider a Hermitian dynamical invariant  $I(t)$  with a discrete spectrum and let  $\{|\lambda_n, a; t\rangle\}$  be a complete set of orthonormal eigenvectors of  $I(t)$ , where  $a \in \{1, 2, \dots, d_n\}$  is a degeneracy label and  $d_n$  is the degree of degeneracy of the eigenvalue  $\lambda_n$ . Then one can show that [10]:

- (1) The eigenvalues  $\lambda_n$  of  $I(t)$  are constant.  
 (2) One can express the evolution operator of the Hamiltonian  $H(t)$  according to

$$U(t) = \sum_n \sum_{a,b=1}^{d_n} u_{ab}^n(t) |\lambda_n, a; t\rangle \langle \lambda_n, b; 0| \tag{2}$$

where  $u_{ab}^n(t)$  are the entries of a unitary matrix  $u^n(t)$  that is determined by the matrix Schrödinger equation:

$$i \frac{d}{dt} u^n(t) = \Delta^n(t) u^n(t) \quad u^n(0) = 1 \tag{3}$$

with

$$\Delta^n(t) := \mathcal{E}^n(t) - \mathcal{A}^n(t) \tag{4}$$

$$\mathcal{E}_{ab}^n := \langle \lambda_n, a; t | H | \lambda_n, b; t \rangle \tag{5}$$

$$\mathcal{A}_{ab}^n := i \langle \lambda_n, a; t | \frac{d}{dt} | \lambda_n, b; t \rangle. \tag{6}$$

For a nondegenerate eigenvalue  $\lambda_n$  of the invariant,  $\mathcal{E}^n(t)$ ,  $\mathcal{A}^n(t)$  and  $\Delta^n(t)$  are scalar functions, and  $u^n(t)$  is a phase factor given by

$$u^n(t) = e^{i\delta_n(t)} e^{i\gamma_n(t)} \tag{7}$$

$$\delta_n(t) := - \int_0^t \mathcal{E}^n(s) ds = - \int_0^t \langle \lambda; s | H(s) | \lambda_n; s \rangle ds \tag{8}$$

$$\gamma_n(t) := \int_0^t \mathcal{A}^n(s) ds = i \int_0^t \langle \lambda; s | \frac{d}{ds} | \lambda_n; s \rangle ds. \tag{9}$$

Furthermore, the state vector

$$|\psi(t)\rangle := u^n(t) |\lambda_n; t\rangle \tag{10}$$

is an exact solution of the Schrödinger equation:

$$i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle. \tag{11}$$

For a  $T$ -periodic invariant, where  $I(t+T) = I(t)$ , the initial state vectors  $|\psi(0)\rangle$  defined by (10) undergo cyclic evolutions, and the phase factors  $e^{i\delta_n(T)}$  and  $e^{i\gamma_n(T)}$  are respectively called the (nonadiabatic) cyclic dynamical and geometric phases [11].

<sup>3</sup> Here ‘nontrivial’ means that  $I(t)$  is not a multiple of the identity operator.

As shown in [10, 12], a similar analysis can be performed for the degenerate eigenvalues  $\lambda_n$ . This yields an expression for the non-Abelian cyclic geometric phase, namely

$$\Gamma^n(T) = \mathcal{T} e^{i \int_0^T \mathcal{A}^n(t) dt} \quad (12)$$

where  $\mathcal{T}$  stands for the time-ordering operator. Noncyclic Abelian and non-Abelian geometric phases can also be treated in terms of dynamical invariants [5, 13].

Finally, we note that in view of equation (2) any dynamical invariant satisfies

$$I(t) = U(t)I(0)U^\dagger(t). \quad (13)$$

### 3. Geometrically equivalent quantum systems

Consider a solution  $|\psi(t)\rangle$  of the Schrödinger equation (11) for the Hamiltonian  $H(t)$ . Then the corresponding pure state  $\Lambda(t)$  may be described by the projection operator  $|\psi(t)\rangle\langle\psi(t)|$ . As time progresses, the state  $\Lambda(t)$  traverses a path in the projective Hilbert space  $\mathcal{P}(\mathcal{H})$ . As we mentioned in section 1, distinct quantum systems may yield the same path  $\Lambda(t)$  in  $\mathcal{P}(\mathcal{H})$  and the same geometric phases [6]<sup>4</sup>.

Next, we suppose that  $|\psi(t)\rangle$  is given by equation (10), i.e. it is an eigenvector of a dynamical invariant  $I(t)$  with a nondegenerate eigenvalue<sup>5</sup>  $\lambda_n$ . Because the information about the geometric phase is entirely included in  $I(t)$ , different Hamiltonians admitting  $I(t)$  as a dynamical invariant would lead to the same set of geometric phases (for the evolution of the eigenstates of  $I(0)$ ). This observation provides the means for a characterization of the GEQs in terms of the symmetries of dynamical invariants.

Specifically, consider another Hamiltonian  $\tilde{H}(t)$  with the same spectral properties as  $H(t)$ , and suppose that  $I(t)$  is an arbitrary Hermitian dynamical invariant of both  $H(t)$  and  $\tilde{H}(t)$ , i.e.  $I(t)$  satisfies equations (1) and

$$\frac{d}{dt} I(t) = i[I(t), \tilde{H}(t)]. \quad (14)$$

Then  $H(t)$  and  $\tilde{H}(t)$  are geometrically equivalent. Next, introduce  $X(t) := \tilde{H}(t) - H(t)$ . In view of equations (1) and (14),

$$[I(t), X(t)] = 0. \quad (15)$$

In other words, the two Hamiltonians  $H(t)$  and  $\tilde{H}(t)$  are related by a symmetry generator  $X(t)$  of the invariant  $I(t)$  according to

$$\tilde{H}(t) = H(t) + X(t). \quad (16)$$

Furthermore, it is not difficult to show that the evolution operator  $\tilde{U}(t)$  of  $\tilde{H}(t)$  may be written in the form

$$\tilde{U}(t) = \sum_n \sum_{a,b=1}^{d_n} \tilde{u}_{ab}^n(t) |\lambda_n, a; t\rangle \langle \lambda_n, b; 0| \quad (17)$$

where  $\tilde{u}^n(t)$  is defined by

$$i \frac{d}{dt} \tilde{u}^n(t) = \tilde{\Delta}^n(t) \tilde{u}^n(t) \quad \tilde{u}(0) = 1 \quad (18)$$

$$\tilde{\Delta}^n(t) := \tilde{\mathcal{E}}^n(t) - \mathcal{A}^n(t) \quad (19)$$

$$\tilde{\mathcal{E}}_{ab}^n := \langle \lambda_n, a; t | \tilde{H} | \lambda_n, b; t \rangle. \quad (20)$$

<sup>4</sup> Note that this is also true for noncyclic geometric phases [13].

<sup>5</sup> Note that this is always possible. That is, for any solution  $|\psi(t)\rangle$ , one can construct a dynamical invariant  $I(t)$  with this property.

Note that the transformation  $H(t) \rightarrow \tilde{H}(t)$  leaves the matrices  $\mathcal{A}^n(t)$  and consequently the geometric phases invariant.

Equations (13), (16) and (17) suggest that, given a time-dependent Hamiltonian  $H(t)$  whose Schrödinger equation is exactly solvable, one can obtain the exact solution of the Schrödinger equation for all the geometrically equivalent Hamiltonians  $\tilde{H}(t)$ . For example, let  $X(t)$  be a polynomial in  $I(t)$ , i.e.

$$X(t) = \sum_{i=1}^p f_i(t)[I(t)]^i \quad (21)$$

where  $f_i$  are arbitrary smooth real-valued functions of  $t$ . Then  $X(t)$  commutes with  $I(t)$  and one obtains a class of exactly solvable time-dependent Hamiltonians given by

$$\tilde{H}(t) = H(t) + \sum_{i=1}^n f_i(t)[I(t)]^i. \quad (22)$$

Next, consider an arbitrary symmetry generator  $X(t)$  of  $I(t)$ , and let

$$Y(t) := U^\dagger(t)X(t)U(t) \quad (23)$$

where  $U(t)$  is the evolution operator for the Hamiltonian  $H(t)$ , i.e. the solution of

$$i \frac{d}{dt} U(t) = H(t)U(t) \quad U(0) = 1. \quad (24)$$

In view of equations (13), (23), (16) and (24), we have

$$[Y(t), I(0)] = 0 \quad (25)$$

$$\tilde{H}(t) = U(t)Y(t)U^\dagger(t) - i\dot{U}(t)U^\dagger(t). \quad (26)$$

These equations indicate that the Hamiltonian  $\tilde{H}(t)$  is related to a symmetry generator of the initial invariant  $I(0)$  through a (canonical) unitary transformation of the Hilbert space [5, 14]:

$$|\phi(t)\rangle \rightarrow |\tilde{\psi}(t)\rangle := U(t)|\phi(t)\rangle \quad (27)$$

where  $|\phi(t)\rangle$  is a solution of the Schrödinger equation with  $Y(t)$  playing the role of the Hamiltonian, i.e.

$$|\phi(t)\rangle = V(t)|\phi(0)\rangle \quad V(t) = \mathcal{T} e^{-i \int_0^t Y(s) ds}. \quad (28)$$

In view of this observation, the evolution operator  $\tilde{U}(t)$  of  $\tilde{H}(t)$  is related to the evolution operator  $V(t)$  of  $Y(t)$  according to

$$\tilde{U}(t) = U(t)V(t). \quad (29)$$

In particular, if  $Y(t)$  is a constant operator,

$$\tilde{U}(t) = U(t)e^{-itY}. \quad (30)$$

Furthermore, if  $X(t)$  is given by equation (21), then in light of (13) and (23),

$$Y(t) = \sum_{i=1}^p f_i(t)[I(0)]^i \quad (31)$$

$$\tilde{U}(t) = U(t)e^{-i \int_0^t Y(s) ds} = U(t)e^{-i \sum_{i=1}^p F_i(t)[I(0)]^i} \quad (32)$$

where  $F_i(t) := \int_0^t f_i(s) ds$ .

Equation (29) provides the desired relationship between the evolution operators of geometrically equivalent Hamiltonians.

In the remainder of this section, we address the question of characterizing the class of all the Hamiltonians that admit a given dynamical invariant  $I(t)$  with the following properties [5, 13]:

- (1)  $I(t)$  is obtained from a Hermitian parametric invariant  $I[\bar{R}]$  as  $I(t) = I[\bar{R}(t)]$ , where  $\bar{R}$  stands for  $(\bar{R}^1, \bar{R}^2, \dots, \bar{R}^d)$ ,  $\bar{R}^i$  are real parameters denoting the coordinates of points of a parameter manifold  $\bar{M}$  and  $\bar{R}(t)$  describes a smooth curve lying in  $\bar{M}$ .
- (2) The spectrum of  $I[\bar{R}]$  is discrete for all  $\bar{R} \in \bar{M}$ .
- (3) In local patches of  $\bar{M}$ , there is a set of orthonormal basis vectors  $|\lambda_n, a; \bar{R}\rangle$  of  $I[\bar{R}]$  that are smooth (single-valued) functions of  $\bar{R}$ .

Clearly one can express the parametric invariant  $I[\bar{R}]$  in the form

$$I[\bar{R}] = \bar{W}[\bar{R}]I_0\bar{W}^\dagger[\bar{R}] \quad (33)$$

where  $\bar{W}[\bar{R}]$  is defined by the condition

$$|\lambda_n, a; \bar{R}\rangle = \bar{W}[\bar{R}]|\lambda_n, a; \bar{R}(0)\rangle \quad (34)$$

and  $I_0 := I[\bar{R}(0)]$ . Now, it is not difficult to check that for any curve  $\bar{R}(t)$  in  $\bar{M}$ , the invariant  $I(t) := I[\bar{R}(t)]$  satisfies the Liouville–von-Neumann equation (1) for the Hamiltonian

$$H_*(t) := i\dot{\bar{W}}[\bar{R}(t)]\bar{W}^\dagger[\bar{R}(t)] \quad (35)$$

where a dot denotes a time derivative. Moreover, any other Hamiltonian that admits the invariant  $I(t)$  is of the form

$$H(t) = H_*(t) + X(t) \quad (36)$$

where  $X(t) := X[\bar{R}(t)]$  and  $X[\bar{R}]$  is any Hermitian operator that commutes with  $I[\bar{R}]$ .

This completes the characterization of the GEQS.

We conclude this section with the following remarks.

- (I) Substituting equations (34) and (35) in equations (5) and (6), we find that for the Hamiltonian  $H_*(t)$

$$\mathcal{E}_{ab}^n(t) = \mathcal{A}_{ab}^n(t) = i\langle \lambda_n, a; \bar{R}(0) | \bar{W}^\dagger[\bar{R}(t)] \dot{\bar{W}}[\bar{R}(t)] | \lambda_n, b; \bar{R}(0) \rangle.$$

Therefore,  $\Delta^n(t) = 0$  and

$$u^n(t) = 1. \quad (37)$$

In particular, the evolution operator for  $H_*(t)$  is given by

$$U_*(t) = \sum_n \sum_{a=1}^{d_n} |\lambda_n, a; t\rangle \langle \lambda_n, a; 0| = W[\bar{R}(t)]. \quad (38)$$

Moreover, for a nondegenerate eigenvalue  $\lambda_n$ , the geometric and dynamical phases cancel each other.

- (II) Let  $Z[\bar{R}]$  be any unitary operator commuting with  $I_0$ . Then the transformation

$$\bar{W}[\bar{R}] \rightarrow \bar{W}'[\bar{R}] := \bar{W}[\bar{R}]Z[\bar{R}] \quad (39)$$

leaves the form of  $I[\bar{R}]$  unchanged. In other words, the operators  $\bar{W}[\bar{R}]$  are subject to ‘gauge transformations’ (39). These transformations are essentially the transformations of the basis vectors of the degeneracy subspaces of the invariant,

$$|\lambda_n, a; \bar{R}\rangle \rightarrow |\lambda_n, a; \bar{R}'\rangle := \bar{W}'[\bar{R}]|\lambda_n, a; \bar{R}(0)\rangle. \quad (40)$$

The physical quantities are invariant under these transformations. Therefore, they may be calculated after making a choice for the gauge:  $\bar{W}[\bar{R}]$ . The choice of  $\bar{W}[\bar{R}]$  is only restricted in the sense that it must satisfy equation (33) and be a single-valued (differentiable) function of  $\bar{R}$ . Note also that the Hamiltonian (35) and its evolution

operator are not invariant under the gauge transformations (39); they transform according to

$$\begin{aligned} H_*(t) &\rightarrow H'_*(t) = H_*(t) + i\bar{W}[\bar{R}(T)]\dot{Z}[\bar{R}(t)]Z[\bar{R}(t)]^\dagger W[\bar{R}(t)]^\dagger \\ U_*(t) &\rightarrow U'_*(t) = U_*(t)Z(t) \end{aligned} \quad (41)$$

respectively. However, it is not difficult to show that the relation (37) survives these transformations.

(III) Suppose that the invariant  $I(t)$  is periodic. Then one may choose a gauge in which

$$\bar{W}[\bar{R}(T)] = \bar{W}[\bar{R}(0)] = 1. \quad (42)$$

This may be realized by requiring that the curve  $\bar{R}(t)$  associated with the invariant is closed, i.e. there is  $T \in \mathbb{R}^+$  such that  $\bar{R}(T) = \bar{R}(0)$ . This in turn implies that  $H_*(T) = H_*(0)$ . Thus the class of GEQs determined by the  $T$ -periodic invariant includes a system with a  $T$ -periodic Hamiltonian  $H_*(t)$ . Furthermore, according to equations (42) and (38),  $U_*(T) = 1$ . This means that the quantum system described by the  $T$ -periodic Hamiltonian  $H_*(t)$  has an evolution loop of period  $T$  [7]. Note, however, that according to equations (41), in an arbitrary gauge,  $H_*(t)$  is not  $T$ -periodic. Yet its evolution operator satisfies  $[U(T), I(0)] = 0$ . This is actually a necessary and sufficient condition for the vectors  $|\lambda_n, a; \bar{R}(0)\rangle$  to perform cyclic evolutions [10].

#### 4. Quantum systems that are geometrically equivalent to a cranked system

By definition, a cranked Hamiltonian has the form

$$H(t) = e^{-iKt} H_0 e^{iKt} \quad (43)$$

where  $K$  and  $H_0$  are constant Hermitian operators. It is not difficult to show that

$$I(t) := H(t) - K = e^{-iKt} (H_0 - K) e^{iKt} \quad (44)$$

is a dynamical invariant for the cranked Hamiltonian (43). Furthermore, one can perform a time-dependent unitary transformation to map the system to a canonically equivalent system with a constant Hamiltonian [14, 15]. This method yields the following expression for the evolution operator of the cranked Hamiltonian (43)<sup>6</sup>:

$$U(t) = e^{-iKt} e^{-i(H_0 - K)t}. \quad (45)$$

Next, we consider the class of Hamiltonians that are geometrically equivalent to a cranked Hamiltonian. According to equations (16) and (44), these Hamiltonians have the following general form:

$$\tilde{H}(t) = e^{-iKt} [H_0 + Y(t)] e^{iKt} \quad (46)$$

where  $Y(t)$  is any Hermitian operator commuting with

$$I_0 = I(0) = H_0 - K. \quad (47)$$

Now, let  $\tilde{Y}(t)$  be any operator commuting with  $I(0)$  and set  $Y(t) = -I(0) + \tilde{Y}(t) = K - H_0 + \tilde{Y}(t)$ . Then equation (46) yields

$$\tilde{H}(t) = e^{-iKt} [K + \tilde{Y}(t)] e^{iKt}. \quad (48)$$

In particular, setting  $\tilde{Y}(t) = 0$ , we find that the cranking operator  $K$  is also geometrically equivalent to the cranked Hamiltonian (43). Note that although  $K$  is time independent, it admits a time-dependent invariant, namely (44). This in turn implies that as the eigenstates

<sup>6</sup> For details see [5].



of this invariant evolve in time (according to the Schrödinger equation defined by  $K$ ) they develop nontrivial (cyclic and noncyclic) geometric phases. Therefore, the above construction provides a simple example of a time-independent Hamiltonian leading to nontrivial geometric phases [4, 7, 16].

Furthermore, we can use an argument similar to the one leading to equation (29) to express the evolution operator  $\tilde{U}(t)$  of  $\tilde{H}(t)$  in terms of the evolution operator  $e^{-iKt}$  of  $K$ . This yields

$$\tilde{U}(t) = e^{-iKt} \mathcal{T} e^{-i \int_0^t \tilde{Y}(s) ds}. \quad (49)$$

Next, we compute geometric phases associated with the eigenstates of the invariant (44). This requires the computation of the eigenvectors  $|\lambda_n, a; t\rangle$  or alternatively the unitary operators  $\tilde{W}[\tilde{R}(t)]$ . We wish to emphasize that one must refrain from identifying the operator  $\tilde{W}[\tilde{R}(t)]$  of equation (33) with  $e^{-iKt}$ . In order to obtain  $\tilde{W}[\tilde{R}(t)]$ , one must first determine the parameter space of the invariant and the single-valued functions  $\tilde{W}[\tilde{R}]$ . In general, the choice  $\tilde{W}[\tilde{R}(t)] = e^{-iKt}$  violates the requirement that  $\tilde{W}[\tilde{R}]$  must be single valued. In view of equations (33) and (44), we can however write

$$\tilde{W}[\tilde{R}(t)] = e^{-iKt} Z(t) \quad (50)$$

where  $Z(t)$  is a unitary operator commuting with  $I(0)$ .

Inserting equation (50) in (35) and (34), we obtain

$$H_*(t) = K + ie^{-iKt} \dot{Z}(t) Z^\dagger(t) e^{iKt} \quad (51)$$

$$|\lambda_n, a; t\rangle = e^{-iKt} Z(t) |\lambda_n, a; 0\rangle. \quad (52)$$

Because  $Z(t)$  and  $\tilde{Y}(t)$  commute with  $I(0)$ , there exist scalar complex-valued functions  $z_n^{ab}$  and  $\tilde{y}_n^{ab}$  satisfying

$$Z(t) |\lambda_n, a; 0\rangle = \sum_{b=1}^{d_n} z_n^{ba}(t) |\lambda_n, b; 0\rangle \quad \tilde{Y}(t) |\lambda_n, a; 0\rangle = \sum_{b=1}^{d_n} \tilde{y}_n^{ba}(t) |\lambda_n, b; 0\rangle. \quad (53)$$

Now, we are in a position to compute the matrices  $\mathcal{E}^n$ ,  $\mathcal{A}^n$  and  $\Delta^n$  for the Hamiltonian (48). The result is

$$\mathcal{E}^n(t) = Z_n^\dagger(t) K_n Z_n(t) + Z_n^\dagger(t) \tilde{Y}_n(t) Z_n(t) \quad (54)$$

$$\mathcal{A}^n(t) = Z_n^\dagger(t) K_n Z_n(t) + i Z_n^\dagger(t) \dot{Z}_n(t) \quad (55)$$

$$\Delta^n(t) = Z_n^\dagger(t) \tilde{Y}_n(t) Z_n(t) - i Z_n^\dagger(t) \dot{Z}_n(t) \quad (56)$$

where  $Z_n$ ,  $K_n$  and  $\tilde{Y}_n$  are  $d_n \times d_n$  matrices with entries  $z_n^{ab}$ ,  $\langle \lambda_n, a; 0 | K | \lambda_n, b; 0 \rangle$ , and  $\tilde{y}_n^{ab}$ , respectively.

Note that  $\Delta^n$  is related to  $\tilde{Y}_n$  by a time-dependent (canonical) unitary transformation. This, in particular, implies

$$u^n(t) = Z_n^\dagger(t) \mathcal{T} e^{-i \int_0^t \tilde{Y}_n(s) ds}. \quad (57)$$

Substituting equations (52) and (57) into (2), we recover equation (49).

For a nondegenerate eigenvalue  $\lambda_n$ , we have  $Z_n(t) = e^{-i\zeta_n(t)}$ , where  $\zeta_n(t) \in \mathbb{R}$  and

$$\mathcal{E}^n(t) = K_n + \tilde{Y}_n(t) \quad \mathcal{A}^n(t) = K_n + \dot{\zeta}_n(t) \quad u^n(t) = e^{i\zeta_n(t)} e^{-i \int_0^t \tilde{Y}_n(s) ds}.$$

In particular, if  $I(t)$  is  $T$ -periodic, the cyclic geometric and dynamical phase angles are given by

$$\gamma_n(T) = K_n T + \zeta_n(T) \quad \delta_n(T) = -K_n T - \int_0^T \tilde{Y}_n(t) dt.$$

### 5. Cyclic states and geometric phases for a time-independent simple harmonic oscillator

Consider the cranked Hamiltonian (43) defined by the initial Hamiltonian

$$H_0 := \frac{p^2}{2M} + \frac{M\Omega^2}{2} x^2 \tag{58}$$

and the cranking operator

$$K := \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 \tag{59}$$

where  $M, \Omega, m, \omega$  are positive real numbers satisfying

$$m > M \quad \text{and} \quad M\Omega^2 > m\omega^2. \tag{60}$$

It is not difficult to compute the cranked Hamiltonian (43). First we use the Backer–Campbell–Hausdorff formula to establish the identities

$$e^{-iKt} x e^{iKt} = \cos(\omega t) x - (m\omega)^{-1} \sin(\omega t) p \tag{61}$$

$$e^{-iKt} p e^{iKt} = m\omega \sin(\omega t) x + \cos(\omega t) p. \tag{62}$$

Substituting equations (58) and (59) into (43) and making use of equations (61) and (62), we have

$$H(t) = \frac{1}{2} \{ [a + b \cos(2\omega t)] p^2 + [c \sin(2\omega t)] (xp + px) + [d + e \cos(2\omega t)] x^2 \} \tag{63}$$

where

$$\begin{aligned} a &:= \frac{1 + v^2}{2M} & b &:= \frac{1 - v^2}{2M} & c &:= \frac{m\omega(1 - v^2)}{2M} \\ d &:= \frac{(m\omega)^2(1 + v^2)}{2M} & e &:= -\frac{(m\omega)^2(1 - v^2)}{2M} & v &:= \frac{M\Omega}{m\omega}. \end{aligned}$$

As seen from equation (63),  $H(t)$  is the Hamiltonian for a periodic time-dependent generalized harmonic oscillator of period  $T = \tau/2$  where  $\tau := 2\pi/\omega$  is the characteristic period of the harmonic oscillator described by the Hamiltonian (59). According to equation (44), the Hamiltonian (63) admits a dynamical invariant of the form

$$\begin{aligned} I(t) = H(t) - K = \frac{1}{2} \{ [a - m^{-1} + b \cos(2\omega t)] p^2 \\ + [c \sin(2\omega t)] (xp + px) + [d - m\omega^2 + e \cos(2\omega t)] x^2 \} \end{aligned} \tag{64}$$

which is also  $T$ -period. Furthermore, in view of the results of the preceding section,  $I(t)$  is also a dynamical invariant for the Hamiltonian  $K$ . In other words, we have constructed a nontrivial  $T$ -periodic dynamical invariant (64) for a time-independent simple harmonic oscillator with arbitrary mass  $m$  and frequency  $\omega$ .

By construction, the eigenstates of the initial invariant  $I(0) = H_0 - K$  perform exact cyclic evolutions of period  $T = \tau/2$ . Note that by virtue of conditions (60), we can identify  $I(0)$  with the Hamiltonian of a simple Harmonic oscillator of mass  $\tilde{m} := (M^{-1} - m^{-1})^{-1}$  and frequency  $\tilde{\omega} := \sqrt{(M^{-1} - m^{-1})(M\Omega^2 - m\omega^2)}$ . Therefore, we can easily obtain the expression for its eigenvectors [17]:

$$|\lambda_n; 0\rangle = (n!)^{-1/2} \tilde{a}^{\dagger n} |0\rangle$$

where

$$\tilde{a} := \sqrt{\frac{\tilde{m}\tilde{\omega}}{2}} \left( x + \frac{ip}{\tilde{m}\tilde{\omega}} \right) \quad \langle x|0\rangle = \left( \frac{\tilde{m}\tilde{\omega}}{\pi} \right)^{1/4} e^{-\tilde{m}\tilde{\omega}x^2/2}. \tag{65}$$

Note that  $|\lambda_n; 0\rangle$  are not eigenvectors of the Hamiltonian  $K$ . Because  $K$  does not depend on time, the corresponding evolution operator is given by  $\mathcal{U}(t) = e^{-iKt}$ . Therefore, the initial eigenvectors  $|\lambda_n; 0\rangle$  evolve according to

$$|\psi_n(t)\rangle = e^{-iKt}|\lambda_n; 0\rangle. \quad (66)$$

Because  $|\lambda_n; 0\rangle$  are eigenvectors of  $I(0)$ , the corresponding pure states perform cyclic evolutions of period  $T$ ;

$$|\lambda_n; T\rangle\langle\lambda_n; T| = |\lambda_n; 0\rangle\langle\lambda_n; 0|.$$

Hence  $|\lambda_n; 0\rangle$  form a complete orthonormal set of basis vectors of the Hilbert space that undergo (nonstationary) cyclic evolutions of period  $T = \tau/2 = \pi/\omega$ .

In order to compute the cyclic geometric and dynamical phases associated with these cyclic states, we need to determine a complete set of orthonormal basis vectors  $|\lambda_n; t\rangle$  of  $I(t)$ . We first note that the harmonic oscillator (59) does not have an evolution loop of period  $T$ . The period of the evolution loops of a time-independent simple harmonic oscillator is an integer multiple  $n\tau$  of its characteristic period  $\tau$ , where  $n$  is a positive integer and  $U(n\tau) = (-1)^n$ , [18]. In particular,  $e^{-iKT} = e^{-iK\tau/2}$  is not a multiple of the identity operator, and  $e^{-itK}$  is not  $T$ -periodic. This is an indication that the operator  $Z(t)$  of equation (50) is different from the identity operator. In order to determine  $Z(t)$  or alternatively the single-valued unitary operators  $\bar{W}[\bar{R}]$ , we need to investigate the parameter space of the corresponding parametric invariant.

We first introduce

$$I[\bar{R}] = \bar{b} \sum_{i=1}^n \bar{R}^i K_i \quad (67)$$

where  $\bar{b}$  is a constant,  $\bar{R} = (\bar{R}^1, \bar{R}^2, \bar{R}^3)$  are parameters of the invariant and

$$K_1 := \frac{1}{4}(X^2 - P^2) \quad (68)$$

$$K_2 := -\frac{1}{4}(XP + PX) = -\frac{1}{4}(xp + px) \quad (69)$$

$$K_3 := \frac{1}{4}(X^2 + P^2) \quad (70)$$

$$X := \sqrt{\tilde{m}\tilde{\omega}}x \quad P := \frac{p}{\sqrt{\tilde{m}\tilde{\omega}}}. \quad (71)$$

Note that  $K_i$  are the generators of the group  $SU(1, 1)$ ; i.e., they satisfy

$$[K_1, K_2] = -iK_3 \quad [K_2, K_3] = iK_1 \quad [K_3, K_1] = -iK_2. \quad (72)$$

The parameter space of the invariant (67) is the unit hyperboloid<sup>7</sup>:

$$\{\bar{R} \in \mathbb{R}^3 \mid -(\bar{R}^1)^2 - (\bar{R}^2)^2 + (\bar{R}^3)^2 = 1, \bar{R}^3 > 0\}.$$

It is convenient to express  $I[\bar{R}]$  in the hyperbolic coordinates

$$\bar{\theta} = \cosh^{-1}(\bar{R}^3) \in \mathbb{R} \quad \bar{\varphi} = \tan^{-1}(\bar{R}^2/\bar{R}^1) \in [0, 2\pi). \quad (73)$$

This yields

$$I[\bar{R}] = I[\bar{\theta}, \bar{\varphi}] = b(\sinh \bar{\theta} \cos \bar{\varphi} K_1 + \sinh \bar{\theta} \sin \bar{\varphi} K_2 + \cosh \bar{\theta} K_3). \quad (74)$$

Next, we introduce

$$\varphi(t) := 2\omega t \quad (75)$$

<sup>7</sup> For a detailed treatment of the parametrization of the quadratic invariants of harmonic oscillators and related issues, see [5].

and write the invariant (64) in the form  $I[\bar{R}(t)]$ , where

$$\bar{b} = 2\tilde{\omega} \quad \bar{R}^1(t) = \frac{1}{2} \left( \tilde{m}b - \frac{e}{\tilde{m}\tilde{\omega}^2} \right) [1 - \cos \varphi(t)] \tag{76}$$

$$\bar{R}^2(t) = -\left(\frac{c}{\tilde{\omega}}\right) \sin \varphi(t) \quad \bar{R}^3(t) = 1 - \frac{1}{2} \left( \tilde{m}b + \frac{e}{\tilde{m}\tilde{\omega}^2} \right) [1 - \cos \varphi(t)]. \tag{77}$$

In view of equations (73), (76) and (77), we have

$$\cosh \bar{\theta}(t) = 1 + \zeta [1 - \cos \varphi(t)] \quad \tan \bar{\varphi}(t) = \frac{\xi \sin \varphi(t)}{1 - \cos \varphi(t)} \tag{78}$$

where

$$\begin{aligned} \zeta &:= -\frac{1}{2} \left( \tilde{m}b + \frac{e}{\tilde{m}\tilde{\omega}^2} \right) = -\frac{(1 - v^2)(1 - \mu^2)}{4(1 - \frac{M}{m})} \\ \xi &:= -\frac{2c}{\tilde{m}\tilde{\omega}b - \frac{e}{\tilde{m}\tilde{\omega}}} = -\frac{2\mu}{1 + \mu^2} \\ \mu &:= \frac{m\omega}{\tilde{m}\tilde{\omega}}. \end{aligned}$$

Now, following the same method as used in [4, 5, 19] to compute the eigenvectors of the dipole Hamiltonian and the generalized harmonic oscillator, we use the commutation relations (72) to express the invariant (74) in the form (33) with

$$\bar{W}[\bar{R}] = e^{-i\bar{\varphi}K_3} e^{-i\bar{\theta}K_2} e^{i\bar{\varphi}K_3}. \tag{79}$$

One can easily check the validity of equation (33) for this choice of  $\bar{W}[\bar{R}]$  by noting that

$$\bar{W}[\bar{R}]K_3\bar{W}[\bar{R}]^\dagger = \sinh \bar{\theta} \cos \bar{\varphi}K_1 + \sinh \bar{\theta} \sin \bar{\varphi}K_2 + \cosh \bar{\theta}K_3.$$

This equation follows from the Baker–Campbell–Hausdorff formula and equations (72).

Having obtained the unitary operator  $\bar{W}[\bar{R}]$ , we may compute  $|\lambda_n; t\rangle = \bar{W}[\bar{R}(t)]|\lambda_n; 0\rangle$  and the corresponding phase angles  $\delta_n(t)$  and  $\gamma_n(t)$ .

First, we note that by construction

$$|\psi_n(t)\rangle = e^{i\delta_n(t)} e^{i\gamma_n(t)} |\lambda_n; t\rangle. \tag{80}$$

This implies that

$$\mathcal{E}^n(t) = \langle \lambda_n; t | K | \lambda_n; t \rangle = \langle \psi_n(t) | K | \psi_n(t) \rangle = \langle \lambda_n; 0 | K | \lambda_n; 0 \rangle \tag{81}$$

where we have also made use of equation (66). We can easily compute the right-hand side of equation (81) using the well known identities [20]:

$$\langle \lambda_n; 0 | x^2 | \lambda_n; 0 \rangle = (\tilde{m}\tilde{\omega})^{-1} (n + \frac{1}{2}) \quad \langle \lambda_n; 0 | p^2 | \lambda_n; 0 \rangle = \tilde{m}\tilde{\omega} (n + \frac{1}{2}).$$

Substituting these equations into (81) and using (8), we obtain

$$\delta_n(t) = \delta_0(t)(2n + 1) \quad \delta_0(t) = -\frac{1}{4} (\mu + \mu^{-1})\omega t. \tag{82}$$

In particular, the dynamical phase angle is given by

$$\delta_n(T) = -\frac{\pi}{4} (\mu + \mu^{-1})(2n + 1).$$

Next, we compute

$$\begin{aligned} \mathcal{A}^n(t) &= i \langle \lambda_n; t | \frac{d}{dt} | \lambda_n; t \rangle = i \langle \lambda_n; 0 | \bar{W}[\bar{R}(t)]^\dagger \dot{\bar{W}}[\bar{R}(t)] | \lambda_n; 0 \rangle \\ &= \frac{1}{4} (2n + 1) [\cosh \bar{\theta}(t) - 1] \dot{\bar{\varphi}}(t). \end{aligned} \tag{83}$$

In the derivation of equation (83), we have employed the identities

$$\begin{aligned}\bar{W}[\bar{R}(t)]^\dagger \dot{\bar{W}}[\bar{R}(t)] &= i[\sinh \bar{\theta}(\cos \bar{\varphi} K_1 + \sin \bar{\varphi} K_2)(1 - \cosh \bar{\theta})K_3] \dot{\bar{\varphi}} \\ &\quad + i(\sin \bar{\varphi} K_1 - \cos \bar{\varphi} K_2) \dot{\bar{\theta}} \\ \langle \lambda_n; 0 | K_1 | \lambda_n; 0 \rangle &= \langle \lambda_n; 0 | K_2 | \lambda_n; 0 \rangle = 0 \\ \langle \lambda_n; 0 | K_3 | \lambda_n; 0 \rangle &= \frac{1}{4} (2n + 1).\end{aligned}$$

Inserting equation (83) into equation (9) and making use of equation (75), we find

$$\begin{aligned}\gamma_n(t) &= \frac{1}{4} (2n + 1) \int_0^t [\cosh \bar{\theta}(t) - 1] \dot{\bar{\varphi}}(t) dt \\ &= \frac{1}{4} (2n + 1) \int_0^{\varphi(t)} [\cosh \bar{\theta}(\varphi) - 1] \frac{d\bar{\varphi}(\varphi)}{d\varphi} d\varphi.\end{aligned}\quad (84)$$

Now, we use the second equation in (78) to calculate

$$\frac{d\bar{\varphi}(\varphi)}{d\varphi} = -\frac{\xi}{(\xi^2 + 1) + (\xi^2 - 1) \cos \varphi}.$$

Substituting this equation in (84) and performing the integral, we finally obtain

$$\begin{aligned}\gamma_n(t) &= (2n + 1)\gamma_0(t) & \gamma_0(t) &= \frac{\zeta \xi \sigma(t)}{4(1 - \xi^2)} = \frac{\mu(1 + \mu^2)(1 - v^2)\sigma(t)}{8(1 - \frac{M}{m})(1 - \mu^2)} \\ \sigma(t) &:= -\varphi(t) + 2|\xi| \tan^{-1} \left[ \frac{\tan \frac{\varphi(t)}{2}}{|\xi|} \right] & &= -2\omega t + 2|\xi| \tan^{-1} \left[ \frac{\tan(\omega t)}{|\xi|} \right].\end{aligned}$$

In particular, the cyclic geometric phase angle associated with the initial state vector  $|\lambda_n; 0\rangle$  has the form

$$\gamma_n(T) = \frac{\pi \zeta \xi (2n + 1)}{2(\xi^2 - 1)} = \frac{\pi \mu(1 + \mu^2)(1 - v^2)(2n + 1)}{4(1 - \frac{M}{m})(\mu^2 - 1)}.\quad (85)$$

Next, we return to the class of GEQs that include the simple harmonic oscillator Hamiltonian (59). The Hamiltonian for such a system is given by equation (48). If we set  $\tilde{Y}(t) = f(t)[H_0 - K]$  in this equation, where  $f$  is an arbitrary smooth positive real-valued function of time, we find a class of time-dependent generalized harmonic oscillator Hamiltonians of the form

$$H_{\text{GHO}}(t) = f(t)H(t) + [1 - f(t)]K\quad (86)$$

with  $H(t)$  given by equation (63). Note that the Hamiltonian  $H(t)$  involves four parameters, namely  $m$ ,  $M$ ,  $\omega$  and  $\Omega$ , that must obey conditions (60). Therefore, equation (86) determines a five-parameter family of time-dependent generalized harmonic oscillators, where four of the parameters are real numbers and the fifth parameter is a function  $f(t)$ . In view of equation (49), the evolution operator for the Hamiltonian (86) is given by

$$\tilde{U}(t) = e^{-iKt} e^{-iF(t)(H_0 - K)}$$

where  $F(t) := \int_0^t f(s) ds$ . Note that, in general,  $H_{\text{GHO}}(t)$  is not  $T$ -periodic, yet it admits a  $T$ -periodic invariant, namely (64).

## 6. Conclusion

We have investigated the quantum systems that give rise to the same set of geometric phases for a complete set of initial state vectors. We termed these systems geometrically equivalent. We

argued that these systems admit a common dynamical invariant and used this observation to yield a complete characterization of these systems. Furthermore, we showed how the evolution operators of the geometrically equivalent systems are related. In particular, we investigated the class of cranked Hamiltonians and applied our general results to simple harmonic oscillators.

We addressed the characterization problem for the systems that are geometrically equivalent to a time-independent simple harmonic oscillator. Our solution showed that this simple system admits periodic dynamical invariants. We used such an invariant to construct a complete orthonormal set of initial state vectors that undergo nonstationary cyclic evolutions. These states involve nontrivial geometric and dynamical phases.

The invariant (64) that we constructed for the simple harmonic oscillator (59) may be put in the form [9]

$$I(t) = \frac{1}{2}[(\rho p - m\dot{\rho}x)^2 + \rho^{-2}x^2]$$

where

$$\rho := \sqrt{\tilde{m}^{-1} - b[1 - \cos(2\omega t)]} \quad (87)$$

satisfies the Ermakov equation [21]

$$\ddot{\rho} + \omega^2\rho = \frac{\eta}{\rho^3} \quad (88)$$

with  $\eta := \tilde{m}^{-1}(\tilde{m}^{-1} - 2b)\omega^2$ . This is indeed to be expected since the general solution of equation (88) may be written [22] in the form  $\rho = \sqrt{c_1x_1^2(t) + c_2x_2^2(t)}$  where  $c_1$  and  $c_2$  are constants and  $x_1$  and  $x_2$  are two linearly independent solutions of the classical equation of motion  $\ddot{x} + \omega^2x = 0$ . Clearly,  $\rho$  as given by equation (87) may be expressed in this form with  $c_1 = \tilde{m}^{-1} - 2b$ ,  $c_2 = \tilde{m}^{-1}$ ,  $x_1 = \sin(\omega t)$  and  $x_2 = \cos(\omega t)$ .

We conclude this article with the following remarks.

- (1) Our results on the cranked Hamiltonian can be easily generalized to the Hamiltonians of the form

$$H(t) = h(t)e^{-ig(t)K} H_0 e^{ig(t)K} \quad (89)$$

where  $g$  and  $h$  are real-valued functions and  $K$  and  $H_0$  are Hermitian operators.

- (2) The analogy between the generalized harmonic oscillator and the interaction of a spinning particle with a changing magnetic field [5] suggest that we can repeat our analysis for the latter system and construct nonstationary cyclic states even for the case of constant magnetic fields.

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